ANALYTICAL SOLUTION OF ONE-DIMENSIONAL BAY FORCED BY SEA BREEZE

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ABSTRACT: This paper introduces a closed-form analytical solution of the one-dimensional (1D), depth-averaged linearized momentum and continuity equations that incorporates linear bottom friction and the non-linear wind stress. The solution describes wind-forced motion in a 1D basin with horizontal bottom as governed by water depth, basin length, bottom friction coefficient, wind speed, and fundamental frequency of an oscillatory wind. The solution displays in compact form general behavior and dependencies of the physical processes, including generation of wind-induced harmonics of the forcing motion, damping, and resonance. The solution can serve as a benchmark test for numerical models of the shallow-water equations, as well as provide estimates of wind-induced motion in enclosed water bodies.

INTRODUCTION

The water of many bays, estuaries, harbors, lakes, and reservoirs is subjected to forcing associated with a periodic or quasi-periodic wind. Such forcing can be the sea breeze, which has solar diurnal periodicity, or the wind associated with passage of seasonal weather fronts with periods typically varying between 3 and 5 days. The leading-order responses of an enclosed body to a steady wind – set-up, set-down, mean current, and recirculating current – are well known (e.g., Ippen and Harleman 1966). Lesser known are the harmonics induced to the water body by a periodic wind. Militello and Kraus (2001) classified these as *forced harmonics* for motion generated directly by the non-linear wind stress $|W|W$ ($W =$ component of wind speed), in contrast to *response harmonics* generated within the water body through interactions contained in the various other non-linear terms in the equations of motion.

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In warm climates in particular, sea breeze can induce substantial diurnal motion in water bodies. Because wind forcing is a quadratic function of its speed, response harmonics generated by sea breeze are present in the water level and current, in addition to the fundamental forcing frequency (Zetler 1971). Nonlinear interactions within the water body also transfer energy into harmonic frequencies, as shown in numerous studies of tidal motion. In two-dimensional, depth-averaged horizontal flow, the quadratic bottom stress, advection, and nonlinear continuity terms generate response harmonics because they are nonlinear with respect to the current velocity, water-surface elevation, or both.

A central consideration in understanding wind-induced water motion and its harmonics is that a water body is locally forced over its entire surface. In contrast, the tide must propagate from a connection to the ocean and is damped by friction as it traverses the bay or estuary. Thus, a distinction between wind and tide is that wind is a *local* forcing whereas the tide is a *boundary* forcing. The relative strength of terms in the equations of motion is, therefore, different.

The sea breeze fluctuates with a frequency of 1 cpd (cycle per day) that is close to frequencies of the diurnal tidal constituents (K1 O1, S1, and others). Similarly, higher harmonics of the water motion induced by sea breeze (wind harmonics) lie at frequencies near the higher harmonics of the diurnal tidal frequencies. Thus, wind harmonics can be obscured by tidal motion and not easily detected. Conversely, tidal constituents must be calculated carefully if wind harmonics are present because they introduce similar motion not of gravitational origin. In embayments where the tidal amplitude is small, the sea breeze can contribute significantly to the diurnal variance of the water surface and current. This situation is common along the coast of Texas, where the strong predominant southeast wind and sea breeze can dominate the tide in producing setup and setdown in its numerous shallow estuaries and bays (Collier and Hedgpeth 1950). Militello (2000) and Militello and Kraus (2001) examined sea-breeze-induced motion at Baffin Bay, Texas, a large, non-tidal water body. Kraus and Militello (1999) document along-axis oscillations in water level exceeding 0.6 m in response to periodic fronts passing East Matagorda Bay, Texas.

This paper introduces a new closed-form analytical solution of the one-dimensional (1D), depth-averaged linearized momentum and continuity equations that incorporates linear bottom friction and the non-linear wind stress. The analytic solution describes linearized wind-forced motion in a 1D basin with horizontal bottom as governed by water depth, basin length, bottom friction coefficient, wind speed, and fundamental frequency of the oscillatory wind.

ORIGIN OF WIND HARMONICS

For focus of discussion and development of the analytic solution, a spatially uniform oscillatory wind blowing parallel to the x-axis is specified. The wind speed is then given as

$$W = w_0 + w\sin(\sigma t) \tag{1}$$

where w_0 = speed of the steady wind, w = amplitude of the oscillatory wind, and $\sigma = 2\pi/T$, in which T = period of the oscillatory wind. A sinusoidal representation for the wind with T = 24 hr is a reasonable description of sea breeze and is implemented below.

To demonstrate how harmonics are generated through wind forcing, for the special case $W \geq 0$, the quadratic wind velocity is

$$W|W| = W^2 = w_0^2 + \frac{1}{2}w^2 + 2w_0 w \sin(\sigma t) - \frac{1}{2}w^2 \cos(2\sigma t) \tag{2}$$

Eq. 2 contains three forcing components as a steady part, a fundamental diurnal frequency σ, and the first even harmonic (semi-diurnal frequency) 2σ of the fundamental. For a pure oscillatory wind, $w_0 = 0$, and the Fourier expansion of the quadratic wind velocity produced by Eq. (2) is

$$W|W| = w^2 \sum_{j=1}^{\infty} A_j \sin[(2j-1)\sigma t] \tag{3}$$

in which

$$A_j = \frac{-8}{\pi(2j-3)(2j-1)(2j+1)} \tag{4}$$

Eq. 4 shows that harmonic frequencies generated by a pure oscillatory wind are odd multiples of the fundamental frequency. Relative magnitudes of A_2 and A_3 to A_1 are 1/5 and 1/35.

ANALYTICAL SOLUTION

Equations of Motion

For a basin of uniform width and water depth $h >> \eta$ (deviation of water surface from still water), the continuity and momentum equations of depth-averaged motion are

$$\frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x} = 0 \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} + \frac{C_f u|u|}{h} + \frac{\rho_a}{\rho}\frac{C_D W|W|}{h} = 0 \tag{6}$$

where t = time, u = horizontal water velocity, g = acceleration due to gravity, C_f = coefficient of bottom friction, ρ_a and ρ are the densities of air and water, respectively, C_D = wind-drag coefficient, and W = wind velocity. Militello and Kraus (2001) showed by scaling analysis of Eqs. 5 and 6 that the pressure gradient term is of the same order as the wind forcing and bottom friction terms, whereas the inertia and advective terms are 2-3 orders of magnitude smaller than the wind forcing term for the stated conditions.

A 1D basin of length L with vertical walls and uniform still-water depth is considered (Fig. 1), over which an along-axis sinusoidal wind blows with spatial uniformity. The governing equations are linearized, including omission of the advective term (which was shown to be small in the scaling analysis), to allow closed-form solution and to eliminate generation of response harmonics by nonlinear terms. Although it is not the intent to compare the linear and non-linear models, the Lorentz approximation for estimating the value of the linear bottom friction coefficient C_{fL} by the principle of equivalent work (Ippen and Harleman 1966) gives $C_{fL} = (8/(3\pi))\, u_m\, C_f$, where u_m is a representative value of the magnitude of the current. The quantity C_{fL} has dimensions of velocity.

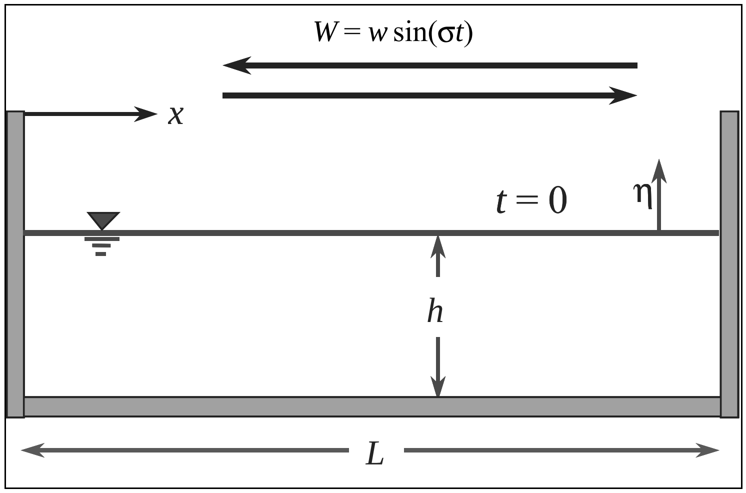

Fig. 1. Sketch of one-dimensional basin with wind forcing, t =0

The continuity and momentum equations (Eqs. 5 and 6) then become

$$\frac{\partial \eta}{\partial t} = -h\frac{\partial u}{\partial x} \tag{7}$$

and

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} - \frac{C_{fL}}{h}u + F \tag{8}$$

where the wind forcing is represented by the function

$$F = F(t) = C_D \frac{\rho_a}{\rho}\frac{W|W|}{h} \tag{9}$$

for pure oscillatory wind specified by Eq. 1 with $w_0 = 0$. Although the wind-drag coefficient varies with the wind speed in some formulations, it is taken to be constant for this derivation, as is C_{fL}.

From Lamb (1945), Ippen and Harleman (1966), and others, linear equation systems such as Eqs. 7 and 8 can be solved by differentiating Eq. 7 with respect to x and Eq. 8 with respect to t, then adding the resultant equations to eliminate η. The one-dimensional inhomogeneous wave equation for u is obtained,

$$u_{tt} + 2du_t - c^2 u_{xx} = F_t \tag{10}$$

in which notation was simplified by defining $d = C_{fL}/(2h)$, and where $c^2 = gh$. The subscripts denote partial differentiation with respect to t and x. The quantity d has the dimensions of frequency, and shows that the friction term in Eq. 10 decreases inversely with the depth.

For the idealized basin, the initial and boundary conditions on u are, respectively, $u(x, 0) = u_t(x, 0) = 0$, and $u(0, t) = u(L, t) = 0$. The water surface is specified to be initially horizontal, and the wind begins blowing at $t = 0$. Symmetry indicates that the problem can be solved over half the basin, for example, on [0, L/2]. In the solution procedure that follows, the full interval [0, L] is chosen as the spatial domain, with symmetry about L/2 for u and anti-symmetry for η serving as checks of the solution.

A solution is sought of the form of a Fourier expansion

$$u(x,t) = \sum_{n=1}^{N} u_n(t) \sin\left[\frac{(2n-1)\pi x}{L}\right] \tag{11}$$

which is a normal-mode equation satisfying the lateral boundary conditions. Substitution of Eq. 11 into Eq. 10 shows that the u_n satisfy the equation describing forced motion with damping,

$$(u_n)_{tt} + 2d(u_n)_t + \sigma_n^2 u_n = \sum_{n=1}^{N} F_n \tag{12}$$

with $\sigma_n = (2n\text{-}1)\pi c/L$ corresponding to odd normal modes. Eqs. 3, 9, and 10 give

$$F_n = \sum_{j=1}^{N} D_{nj} \cos(\sigma_j t) \tag{13}$$

where $\sigma_j = (2j\text{-}1)\,\sigma$ are the frequencies of harmonics forced by the quadratic wind stress and

$$D_{nj} = \frac{4}{(2n-1)\pi} C_D \frac{\rho_a}{\rho} \frac{\sigma}{h} w^2 A_j \tag{14}$$

The solution of Eq. 12 with the initial conditions depends on the relative values of σ_n and d, by which either underdamping ($d < \sigma_n$) or overdamping ($d > \sigma$) can occur. Note that d contains the water depth and that the σ_n will have a wide range if a reasonable number of components (e.g., $N = 7$) is assigned. Critical damping ($d = \sigma_n$) cannot occur in a practical situation for input values specified to one or two significant figures. The formal solution given below for overdamping describes both the under- and overdamping situations for complex arguments of the exponential functions appearing in it.

The solution of the linearized shallow-water wave equations for the basin with an impressed wind blowing as $W = w\sin(\sigma t)$ and with initial conditions of a flat water surface and boundary conditions of zero velocity is found to be, for the depth-averaged velocity,

$$u(x,t) = \sum_{n=1}^{N} \left(\sum_{j=1}^{J} u_{nj} \right) \sin\left[\frac{(2n-1)\pi x}{L}\right] \tag{15}$$

with the real part of

$$u_{nj} = C_{1nj} e^{\lambda_1 t} + C_{2nj} e^{\lambda_2 t} + C_{3nj} \cos\sigma_j t + C_{4nj} \sin\sigma_j t \tag{16}$$

where again, with $d = C_{fL}/(2h)$ and C_{fL} a friction coefficient for linearized bottom stress,

$$\begin{aligned} \lambda_1 &= -d + \sqrt{d^2 - \sigma_n^2} \\ \lambda_2 &= -d - \sqrt{d^2 - \sigma_n^2} \quad , \end{aligned} \tag{17}$$

$$
\begin{aligned}
C_{1nj} &= -\frac{\hat{D}_{nj}}{2\sqrt{d^2-\sigma_n^2}}\left[(\sigma_n^2-\sigma_j^2)\sqrt{d^2-\sigma_n^2}+d(\sigma_n^2+\sigma_j^2)\right] \\
C_{2nj} &= -C_{1nj}-\hat{D}_{nj}(\sigma_n^2-\sigma_j^2) \\
C_{3nj} &= \hat{D}_{nj}(\sigma_n^2-\sigma_j^2) \\
C_{4nj} &= 2d\sigma_j\hat{D}_{nj}
\end{aligned}
\tag{18}
$$

$$
\hat{D}_{nj} = \frac{D_{nj}}{(\sigma_n^2-\sigma_j^2)^2+4d^2\sigma_j^2} \tag{19}
$$

The water-surface elevation is given by integrating Eq. 7 with Eq. 15 for u to give

$$
\eta(x,t) = \sum_{n=1}^{N}(2n-1)\left(\sum_{j=1}^{J}\eta_{nj}\right)\cos\left[\frac{(2n-1)\pi x}{L}\right] \tag{20}
$$

with the real part of the following equation taken:

$$
\eta_{nj} = \frac{C_{1nj}}{\lambda_1}(e^{\lambda_1 t}-1)+\frac{C_{2nj}}{\lambda_2}(e^{\lambda_2 t}-1)+\frac{C_{3nj}}{\sigma_j}\sin\sigma_j t-\frac{C_{4nj}}{\sigma_j}(\cos\sigma_j t-1) \tag{21}
$$

This solution describes linearized wind-forced motion in a 1D basin as governed by five parameters: water depth, basin length, bottom friction coefficient, wind speed, and fundamental frequency of the oscillatory wind. The solution includes the initial transients and possible mixed under-damping ($d < \sigma_n$) and over-damping ($d > \sigma_n$), depending on the normal modes, as can occur according to the values of λ_1 and λ_2.

RESULTS

Example Dynamics of Analytical Solution

For examining properties of the analytic solution, the geometry of an idealized basin was established that approximated Baffin Bay, Texas (Militello and Kraus 2001), as L = 29 km, h = 1 m, and C_{fL} =0.009, upon which a spatially uniform sinusoidal wind was imposed with w = 10 m s^{-1} and C_D = 0.0016. Through trial runs, 3-digit reproducibility was obtained with four wind harmonics (J = 4) and nine normal modes (N = 9).

The time series of η from Eq. 20 at $x = L$ - 500 m and u from Eq. 15 at the middle of the basin are plotted in Fig. 2 for 3 days. Day 0 was omitted to allow transients to disappear. The greatest variation in water level and velocity are experienced at the basin ends and middle, respectively.

The water-surface elevation and current velocity contain complex structure through the presence of both wind-generated harmonics and normal-mode frequencies. The spectra of η and u shown in Fig. 3 indicate strong motion at the fundamental frequency of 1 cpd and energy at the odd forced harmonics associated with the quadratic wind stress. The peak at 4.65 cpd is the first normal (seiching) mode of the basin. The amplitudes of the harmonics can also be obtained from the solutions, and Eqs. 15 and 20.

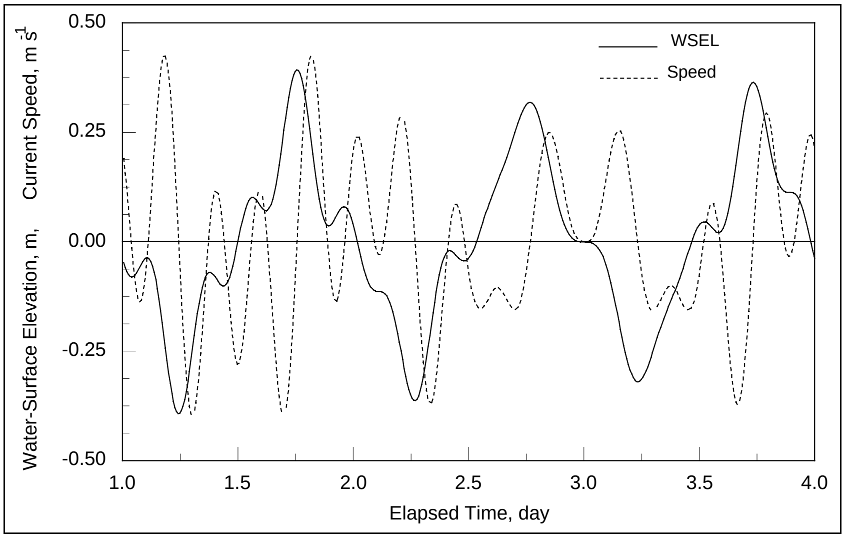

Fig. 2. Calculated time series of water-surface elevation (WSEL) and current, 1-m depth

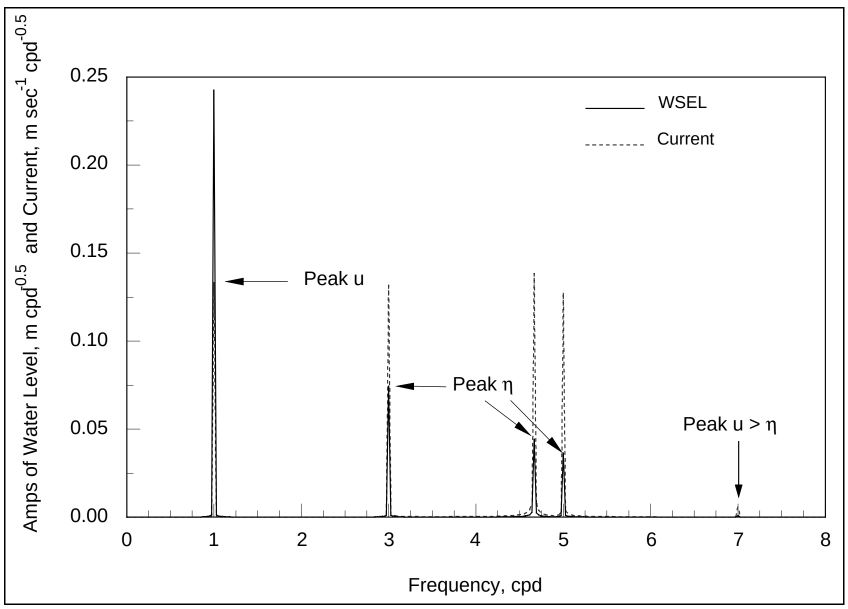

Fig.3. Spectra of water-surface elevation and current, 1-m depth

Figures 4 and 5 respectively plot the water level and velocity along the basin at hourly intervals for the first 12 hr after the start of Day 1. Because the motion of the water level and current are complicated (Fig. 2), changes at a fixed time interval are not regular. The water level fluctuates between about ±0.4 m, and the velocity fluctuates between about 0.4 and -0.3 m s^{-1} for this particular 12-hr interval. At the location $L/2$, η is anti-symmetric and u is symmetric. Also, the water surface along the basin exhibits some curvature, departing from a straight line that might be expected intuitively.

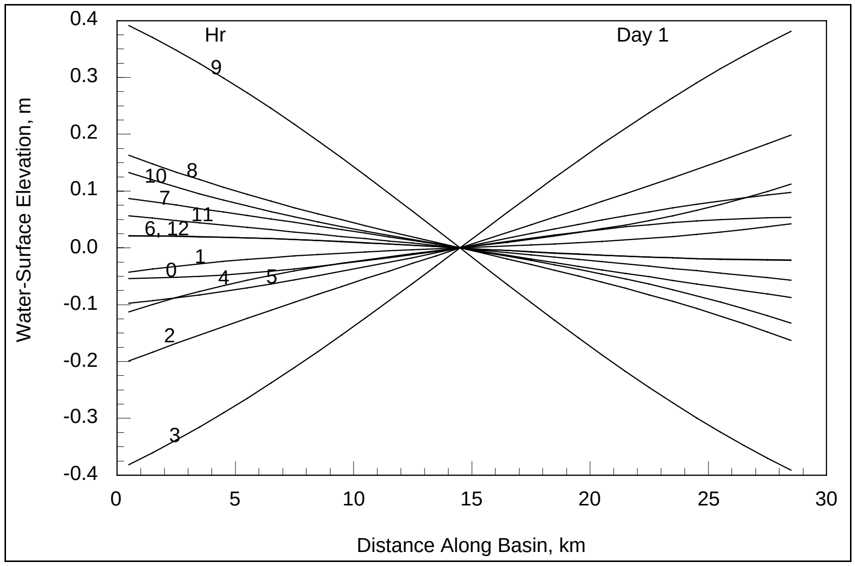

Fig. 4. Selected water levels along the basin, 1-m depth

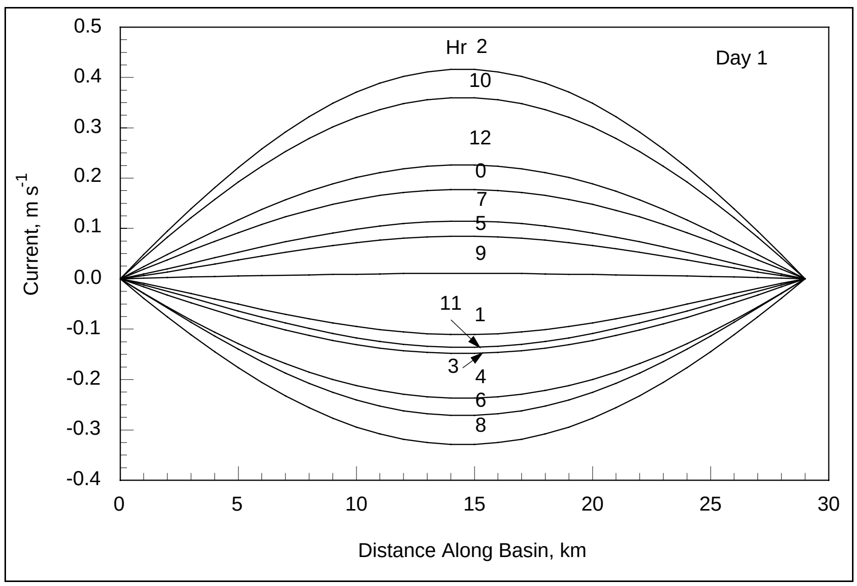

Fig. 5. Selected current velocities along the basin, 1-m depth

To demonstrate other properties of the analytic solution, the depth in the previous example was changed to $h = 2$ m. Time series of the water-surface elevation and current are plotted in Fig. 6 with corresponding spectra in Fig. 7. These are analogous to Figs. 2 and 3 for the situation of $h = 1$ m. The responses (amplitudes) of both the water level and current speed are smaller for the greater depth because the wind must move more water. Also, the first normal mode is now located at 6.60 cpd. As a consequence of the different normal modes, the time series differ for the two ambient depths.

Action of Bottom Friction in Presence of Sea Breeze

With bottom friction acting, higher-mode frequencies damp more than lower modes. Damping of water-surface elevation amplitudes for the friction coefficient C_f ranging from

0 (no friction) to 0.02 (strong damping, as over a porous reef) are shown in Fig. 8 for the idealized basin. Amplitudes are normalized by the no-friction value of the corresponding frequency. Motions on the fundamental forcing frequency (1 cpd), the 1st and 2nd harmonics (3 and 5 cpd), and the first resonant mode (4.7 cpd) are present. The curves for the fundamental and harmonic frequencies indicate steep damping for smaller values of friction coefficients, tapering to mild slopes with greater values of friction. Curves for the harmonic frequencies approach near-zero slope with increased friction coefficient. Motion at these frequencies is present, even at large friction values, because it is forced over the entire surface of the water body. In contrast, the resonant frequency damps to near zero for even small values of friction, being a boundary-induced phenomenon.

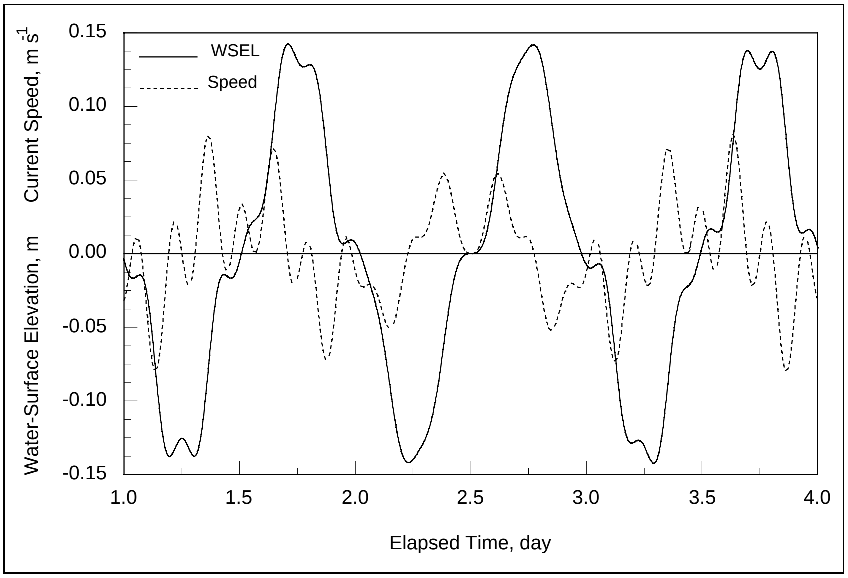

Fig. 6. Time series of water-surface elevation and current, 2-m depth

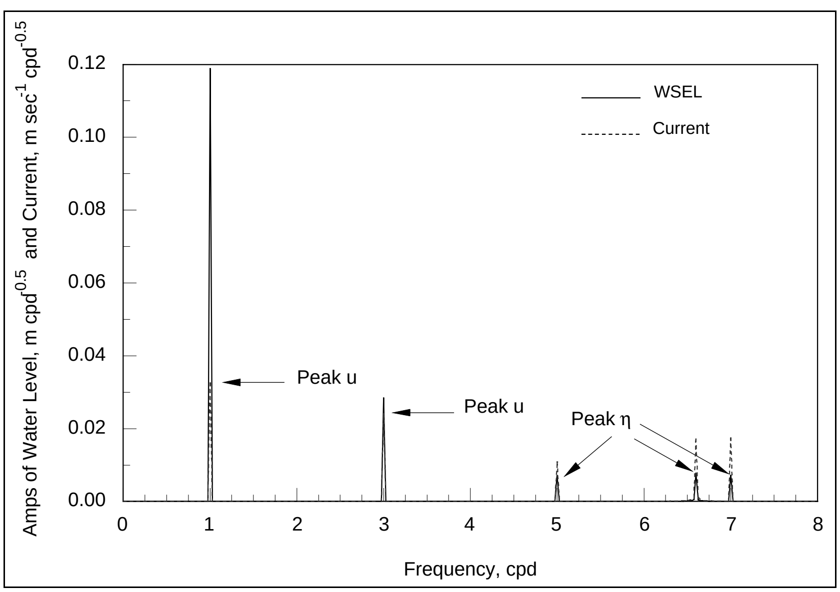

Fig. 7. Spectra of water-surface elevation and current, 2-m depth

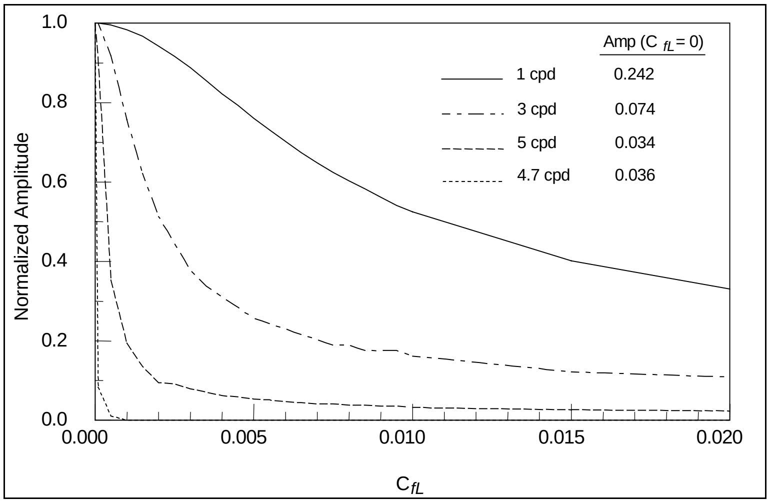

Fig. 8. Normalized amplitudes for oscillatory wind-forced water level for friction coefficient ranging from 0 to 0.020.

CONCLUSION

An analytical solution for an idealized 1D basin was developed to study the response of initially quiescent water to oscillatory wind as governed by the linearized equations of motion with quadratic wind stress. The solution displays in compact form general behavior and dependencies of the physical processes, including generation of harmonics of the motion, damping, and resonance. The solution can serve as a benchmark test for numerical models of the shallow-water equations to examine properties such as numerical damping, generation of spurious motions, symmetry, and accuracy. It also provides a convenient procedure for making first-order estimates of wind-induced motion in enclosed water bodies such as bays, estuaries, lakes, and reservoirs.

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